MA 441, Fall 2013

Assignment 7

This homework is due *Thursday* Oct 17.

There are total 29 points in this assignment. 26 points is considered 100%. If you go over 26 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 3.4–3.5 in Bartle–Sherbert.

(1) [2pt] (3.4.1) Give an example of an unbounded sequence that has a convergent subsequence.

(2) [3pt] (3.4.5) Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the "shuffled" sequence $Z = (z_n)$ be defined by $z_1 = x_1, z_2 = y_2, \cdots, z_{2n-1} = x_n, z_{2n} = y_n, \cdots$. Show that Z converges if and only if both X and Y are convergent and $\lim X = \lim Y$.

(3) [3pt] (3.4.14) Let (x_n) be a bounded sequence and let

$$s = \sup\{x_n : n \in \mathbb{N}\}.$$

Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to s.

(4) (a) [3pt] (Exercise 3.4.9) Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.

(b) [3pt] Suppose that every subsequence of $X = (x_n)$ has a converging subsequence. Is it true that in this case, X must converge?

(5) [2pt] (3.5.1) Give example of a bounded sequence that is not a Cauchy sequence.

(6) [3pt] (3.5.2b) Show directly from the definition that the sequence $\left(1 + \frac{1}{2!} + \ldots + \frac{1}{n!}\right)$ is a Cauchy sequence.

(7) [3pt] (Exercise 3.5.4) Show directly from definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.

(8) [4pt] (3.5.9) If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence. (*Hint:* $x_{n+2} - x_n = (x_{n+2} - x_{n+1}) + (x_{n+1} - x_n)$. Generalize this to $x_{n+m} - x_n$.)

(9) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$. Is it true that if for any $\varepsilon > 0$, there is a natural number $H = H(\varepsilon)$ such that for all $n > H$, $|x_n - x_{n+1}| < \varepsilon$, then X is a Cauchy sequence? (*Hint*: inspect partial sums of harmonic series, or look at $x_n = \sqrt{n}$).